

# DISTRIBUTION OF THE ZETA FUNCTIONS SINGULARITIES FOR COMPACT EVEN-DIMENSIONAL LOCALLY SYMMETRIC SPACES

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ABSTRACT. For compact, even-dimensional, locally symmetric spaces, we obtain precise estimates on the number of singularities of Selberg's and Ruelle's zeta functions considered by U. Bunke and M. Olbrich.

## 1. INTRODUCTION

Let  $Y = \Gamma \backslash G/K = \Gamma \backslash X$  be a compact,  $n$ -dimensional ( $n$  even), locally symmetric Riemannian manifold with negative sectional curvature, where  $G$  is a connected semisimple Lie group of real rank one,  $K$  is a maximal compact subgroup of  $G$  and  $\Gamma$  is a discrete co-compact torsion free subgroup of  $G$ .

We require  $G$  to be linear in order to have complexification available.

In [4], Bunke and Olbrich studied the zeta functions of Selberg and Ruelle associated with a locally homogeneous vector bundle of the unit sphere bundle of  $Y$ .

It is well known that the number of the zeros  $\rho$ ,  $|\rho| \leq t$  of an entire function of order not larger than  $m$  is  $O(t^m)$  (see, [5, p. 510]). The classical Selberg zeta function is an entire function of order two (see, [10]). However, the number of it's zeros on the interval  $\frac{1}{2} + ix$ ,  $0 < x < t$  is  $\frac{A}{4\pi}t^2 + O(t)$ , for some explicitly determined constant  $A$  (see, [6]-[9]). The main purpose of this paper is to give an analogous result for the zeta functions considered in [4].

## 2. PRELIMINARIES

In the sequel we follow the notation of [4] (see also [2]).

Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$ ,  $\mathfrak{a}$  a maximal abelian subspace of  $\mathfrak{p}$  and  $M$  the centralizer of  $\mathfrak{a}$  in  $K$  with Lie algebra  $\mathfrak{m}$ . We normalize the  $\text{Ad}(G)$ -invariant inner product  $(\cdot, \cdot)$  on  $\mathfrak{g}$  to restrict to the metric on  $\mathfrak{p}$ . Let  $SX = G/M$  be the unit sphere bundle of  $X$ . Hence  $SY = \Gamma \backslash G/M$ .

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Let  $\Phi(\mathfrak{g}, \mathfrak{a})$  be the root system and  $W = W(\mathfrak{g}, \mathfrak{a}) \cong \mathbb{Z}_2$  its Weyl group. Fix a system of positive roots  $\Phi^+(\mathfrak{g}, \mathfrak{a}) \subset \Phi(\mathfrak{g}, \mathfrak{a})$ . Let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{n}_\alpha$$

be the sum of the root spaces corresponding to elements of  $\Phi^+(\mathfrak{g}, \mathfrak{a})$ . The decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{n}$  corresponds to the Iwasawa decomposition of the group  $G = KAN$ . Define  $\rho \in \mathfrak{a}_\mathbb{C}^*$  by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{n}_\alpha) \alpha.$$

The positive Weyl chamber  $\mathfrak{a}^+$  is the half line in  $\mathfrak{a}$  on which the positive roots take positive values. Let  $A^+ = \exp(\mathfrak{a}^+) \subset A$ .

The symmetric space  $X$  has a compact dual space  $X_d = G_d/K$ , where  $G_d$  is the analytic subgroup of  $GL(n, \mathbb{C})$  corresponding to  $\mathfrak{g}_d = \mathfrak{k} \oplus \mathfrak{p}_d$ ,  $\mathfrak{p}_d = i\mathfrak{p}$ . We normalize the metric on  $X_d$  in such a way that the multiplication by  $i$  induces an isometry between  $\mathfrak{p}$  and  $\mathfrak{p}_d$ .

Let  $i^* : R(K) \rightarrow R(M)$  be the restriction map induced by the embedding  $i : M \hookrightarrow K$ , where  $R(K)$  and  $R(M)$  are the representation rings over  $\mathbb{Z}$  of  $K$  and  $M$ , respectively.

Since  $n$  is even, every  $\sigma \in \hat{M}$  is invariant under the action of the Weyl group  $W$  (see, [4, p. 27]). Let  $\sigma \in \hat{M}$ . We choose  $\gamma \in R(K)$  such that  $i^*(\gamma) = \sigma$  and represent it by  $\sum a_i \gamma_i$ ,  $a_i \in \mathbb{Z}$ ,  $\gamma_i \in \hat{K}$ . Set

$$V_\gamma^\pm = \sum_{\text{sign}(a_i) = \pm 1} \sum_{m=1}^{|a_i|} V_{\gamma_i},$$

where  $V_{\gamma_i}$  is the representation space of  $\gamma_i$ . Define  $V(\gamma)^\pm = G \times_K V_\gamma^\pm$  and  $V_d(\gamma)^\pm = G_d \times_K V_\gamma^\pm$ . To  $\gamma$  we associate  $\mathbb{Z}_2$ -graded homogeneous vector bundles  $V(\gamma) = V(\gamma)^+ \oplus V(\gamma)^-$  and  $V_d(\gamma) = V_d(\gamma)^+ \oplus V_d(\gamma)^-$  on  $X$  and  $X_d$ , respectively. Let

$$V_{Y, \chi}(\gamma) = \Gamma \backslash (V_\chi \otimes V(\gamma))$$

be a  $\mathbb{Z}_2$ -graded vector bundle on  $Y$ , where  $(\chi, V_\chi)$  is a finite-dimensional unitary representation of  $\Gamma$ .

By [4, p. 36],

$$(2.1) \quad \frac{\chi(Y)}{\chi(X_d)} = (-1)^{\frac{n}{2}} \frac{\text{vol}(Y)}{\text{vol}(X_d)}.$$

Reasoning as in the beginning of Subsection 1.1.2 in [4], we choose a Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{m}$  and a system of positive roots  $\Phi^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t})$ . Then,  $\rho_{\mathfrak{m}} \in \mathfrak{it}^*$ , where

$$\rho_{\mathfrak{m}} = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t})} \alpha.$$

Let  $\mu_{\sigma} \in \mathfrak{it}^*$  be the highest weight of  $\sigma$ . Set

$$c(\sigma) = |\rho|^2 + |\rho_{\mathfrak{m}}|^2 - |\mu_{\sigma} + \rho_{\mathfrak{m}}|^2,$$

where the norms are induced by the complex bilinear extension to  $\mathfrak{g}_{\mathbb{C}}$  of the inner product  $(\cdot, \cdot)$ . Finally, we introduce the operators (see, [4, p. 28])

$$\begin{aligned} A_d(\gamma, \sigma)^2 &= \Omega + c(\sigma) : C^\infty(X_d, V_d(\gamma)) \rightarrow C^\infty(X_d, V_d(\gamma)), \\ A_{Y, \chi}(\gamma, \sigma)^2 &= -\Omega - c(\sigma) : C^\infty(Y, V_{Y, \chi}(\gamma)) \rightarrow C^\infty(Y, V_{Y, \chi}(\gamma)), \end{aligned}$$

$\Omega$  being the Casimir element of the complex universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  of  $\mathfrak{g}$ .

Let  $m_{\chi}(s, \gamma, \sigma) = \text{Tr } E_{A_{Y, \chi}(\gamma, \sigma)}(\{s\})$ ,  $m_d(s, \gamma, \sigma) = \text{Tr } E_{A_d(\gamma, \sigma)}(\{s\})$ , where  $E_A(\cdot)$  denotes the family of spectral projections of a normal operator  $A$ .

Now, we choose a maximal abelian subalgebra  $\mathfrak{t}$  of  $\mathfrak{m}$ . Then,  $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$  is a Cartan subalgebra of  $\mathfrak{g}_{\mathbb{C}}$ . Let  $\Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$  be a positive root system having the property that, for  $\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ ,  $\alpha|_{\mathfrak{a}} \in \Phi^+(\mathfrak{g}, \mathfrak{a})$  implies  $\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ . Let

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \alpha.$$

We set  $\rho_{\mathfrak{m}} = \delta - \rho$ . Define the root vector  $H_{\alpha} \in \mathfrak{a}$  for  $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$  by

$$\lambda(H_{\alpha}) = \frac{(\lambda, \alpha)}{(\alpha, \alpha)},$$

where  $\lambda \in \mathfrak{a}^*$ .

For  $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})$ , we define  $\varepsilon_{\alpha}(\sigma) \in \{0, \frac{1}{2}\}$  by

$$e^{2\pi i \varepsilon_{\alpha}(\sigma)} = \sigma(e^{2\pi i H_{\alpha}}) \in \{\pm 1\}.$$

According to [4, p. 47], the root system  $\Phi^+(\mathfrak{g}, \mathfrak{a})$  is of the form  $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$  or  $\Phi^+(\mathfrak{g}, \mathfrak{a}) = \{\frac{\alpha}{2}, \alpha\}$  for the long root  $\alpha$ . Let  $\alpha$  be the long root in  $\Phi^+(\mathfrak{g}, \mathfrak{a})$ . We set  $T = |\alpha|$ . For  $\sigma \in \hat{M}$ ,  $\epsilon_{\sigma} \in \{0, \frac{1}{2}\}$  is given by

$$\epsilon_{\sigma} \equiv \frac{|\rho|}{T} + \varepsilon_{\alpha}(\sigma) \pmod{\mathbb{Z}}.$$

We define the lattice  $L(\sigma) \subset \mathbb{R} \cong \mathfrak{a}^*$  by  $L(\sigma) = T(\epsilon_\sigma + \mathbb{Z})$ . Finally, for  $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$  we set

$$P_\sigma(\lambda) = \prod_{\beta \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})} \frac{(\lambda + \mu_\sigma + \rho_{\mathfrak{m}}, \beta)}{(\delta, \beta)}.$$

Since  $n$  is even, there exists a  $\sigma$ -admissible  $\gamma \in R(K)$  for every  $\sigma \in \hat{M}$  (see, [4, p. 49, Lemma 1.18]). Here,  $\gamma \in R(K)$  is called  $\sigma$ -admissible if  $i^*(\gamma) = \sigma$  and  $m_d(s, \gamma, \sigma) = P_\sigma(s)$  for all  $0 \leq s \in L(\sigma)$ .

### 3. ZETA FUNCTIONS AND THE GEODESIC FLOW

Since  $\Gamma \subset G$  is co-compact and torsion free, there are only two types of conjugacy classes - the class of the identity  $1 \in \Gamma$  and classes of hyperbolic elements.

Let  $g \in G$  be hyperbolic. Then there is an Iwasawa decomposition  $G = NAK$  such that  $g = am \in A^+M$ . Following [4, p. 59], we define

$$l(g) = |\log(a)|.$$

Let  $\Gamma_h$ , resp.  $\text{P}\Gamma_h$  denote the set of the  $\Gamma$ -conjugacy classes of hyperbolic resp. primitive hyperbolic elements in  $\Gamma$ .

Let  $\varphi$  be the geodesic flow on  $SY$  determined by the metric of  $Y$ . In the representation  $SY = \Gamma \backslash G/M$ ,  $\varphi$  is given by

$$\varphi : \mathbb{R} \times SY \ni (t, \Gamma gM) \rightarrow \Gamma g \exp(-tH)M \in SY,$$

where  $H$  is the unit vector in  $\mathfrak{a}^+$ . If  $V_\chi(\sigma) = \Gamma \backslash (G \times_M V_\sigma \otimes V_\chi)$  is the vector bundle corresponding to finite-dimensional unitary representations  $(\sigma, V_\sigma)$  of  $M$  and  $(\chi, V_\chi)$  of  $\Gamma$ , then we define a lift  $\varphi_{\chi, \sigma}$  of  $\varphi$  to  $V_\chi(\sigma)$  by (see, [4, p. 95])

$$\varphi_{\chi, \sigma} : \mathbb{R} \times V_\chi(\sigma) \ni (t, [g, v \otimes w]) \rightarrow [g \exp(-tH), v \otimes w] \in V_\chi(\sigma).$$

For  $\text{Re}(s) > 2\rho$ , the Ruelle zeta function for the flow  $\varphi_{\chi, \sigma}$  is defined by the infinite product

$$Z_{R, \chi}(s, \sigma) = \prod_{\gamma_0 \in \text{P}\Gamma_h} \det \left( 1 - (\sigma(m) \otimes \chi(\gamma_0)) e^{-sl(\gamma_0)} \right)^{(-1)^{n-1}}.$$

The Selberg zeta function for the flow  $\varphi_{\chi, \sigma}$  is given by

$$(3.1) \quad Z_{S,\chi}(s, \sigma) =$$

$$\prod_{\gamma_0 \in \text{P}\Gamma_{\mathfrak{h}}} \prod_{k=0}^{+\infty} \det \left( 1 - \left( \sigma(m) \otimes \chi(\gamma_0) \otimes S^k(\text{Ad}(ma)_{\bar{\mathfrak{n}}}) \right) e^{-(s+\rho)l(\gamma_0)} \right),$$

for  $\text{Re}(s) > \rho$ , where  $S^k$  denotes the  $k$ -th symmetric power of an endomorphism,  $\bar{\mathfrak{n}} = \theta\mathfrak{n}$  is the sum of negative root spaces of  $\mathfrak{a}$  as usual, and  $\theta$  is the Cartan involution of  $\mathfrak{g}$ .

Let  $\mathfrak{n}_{\mathbb{C}}$  be the complexification of  $\mathfrak{n}$ . For  $\lambda \in \mathbb{C} \cong \mathfrak{a}_{\mathbb{C}}^*$  let  $\mathbb{C}_{\lambda}$  denote the one-dimensional representation of  $A$  given by  $A \ni a \rightarrow a^{\lambda}$ . Let  $p \geq 0$ . There exist sets

$$I_p = \left\{ (\tau, \lambda) \mid \tau \in \hat{M}, \lambda \in \mathbb{R} \right\}$$

such that  $\Lambda^p \mathfrak{n}_{\mathbb{C}}$  as a representation of  $MA$  decomposes with respect to  $MA$  as

$$\Lambda^p \mathfrak{n}_{\mathbb{C}} = \sum_{(\tau, \lambda) \in I_p} V_{\tau} \otimes \mathbb{C}_{\lambda},$$

where  $V_{\tau}$  is the space of the representation  $\tau$ . Bunke and Olbrich proved that the Ruelle zeta function  $Z_{R,\chi}(s, \sigma)$  has the following representation (see, [4, p. 99, Prop. 3.4])

$$(3.2) \quad Z_{R,\chi}(s, \sigma) = \prod_{p=0}^{n-1} \prod_{(\tau, \lambda) \in I_p} Z_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)^{(-1)^p}.$$

Let  $d_Y = -(-1)^{\frac{n}{2}}$ . The following theorem holds true (see, [4, p. 113, Th. 3.15])

**Theorem A.** *The Selberg zeta function  $Z_{S,\chi}(s, \sigma)$  has a meromorphic continuation to all of  $\mathbb{C}$ . If  $\gamma$  is  $\sigma$ -admissible, then the singularities (zeros and poles) of  $Z_{S,\chi}(s, \sigma)$  are the following ones:*

- (1) *at  $\pm is$  of order  $m_{\chi}(s, \gamma, \sigma)$  if  $s \neq 0$  is an eigenvalue of  $A_{Y,\chi}(\gamma, \sigma)$ ,*
- (2) *at  $s = 0$  of order  $2m_{\chi}(0, \gamma, \sigma)$  if  $0$  is an eigenvalue of  $A_{Y,\chi}(\gamma, \sigma)$ ,*
- (3) *at  $-s$ ,  $s \in T(\mathbb{N} - \epsilon_{\sigma})$  of order  $2 \frac{d_Y \dim(\chi) \text{vol}(Y)}{\text{vol}(X_d)} m_d(s, \gamma, \sigma)$ . Then  $s > 0$  is an eigenvalue of  $A_d(\gamma, \sigma)$ .*

*If two such points coincide, then the orders add up.*

Since  $\lambda$  runs through  $\mathfrak{a}$ -weights on the exterior algebra of  $\mathfrak{n}$ , the shifts  $\rho - \lambda$  that appear in (3.2) are always contained in the interval  $[-\rho, \rho]$ . Hence, by (3.2) and Theorem A,  $s_R \in \mathbb{R}$  for all singularities  $s_R$  of the  $Z_{R,\chi}(s, \sigma)$  with  $|\operatorname{Re}(s_R)| > \rho$ .

In [2], we proved that there exist entire functions  $Z_S^1(s)$ ,  $Z_S^2(s)$  of order at most  $n$  such that

$$(3.3) \quad Z_{S,\chi}(s, \sigma) = \frac{Z_S^1(s)}{Z_S^2(s)}.$$

Here,  $\gamma$  is  $\sigma$ -admissible, the zeros of  $Z_S^1(s)$  correspond to the zeros of  $Z_{S,\chi}(s, \sigma)$  and the zeros of  $Z_S^2(s)$  correspond to the poles of  $Z_{S,\chi}(s, \sigma)$ . The orders of the zeros of  $Z_S^1(s)$  resp.  $Z_S^2(s)$  equal the orders of the corresponding zeros resp. poles of  $Z_{S,\chi}(s, \sigma)$ .

#### 4. RESULTS

**Lemma 4.1.** *If  $\gamma$  is  $\sigma$ -admissible, then*

$$Z_{S,\chi}(s, \sigma) = e^{-K \int_0^s P_\sigma(w) \left\{ \begin{array}{ll} \tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = \frac{1}{2} \\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = 0 \end{array} \right\} dw} Z_{S,\chi}(-s, \sigma),$$

where  $K = \frac{2\pi \dim(\chi)\chi(Y)}{\chi(X_d)T}$ .

*Proof.* By [4, p. 118, Th. 3.19],  $Z_{S,\chi}(s, \sigma)$  has the representation

$$Z_{S,\chi}(s, \sigma) = \det \left( A_{Y,\chi}(\gamma, \sigma)^2 + s^2 \right) \det(A_d(\gamma, \sigma) + s)^{-\frac{2 \dim(\chi)\chi(Y)}{\chi(X_d)}} \cdot \exp \left( \frac{\dim(\chi)\chi(Y)}{\chi(X_d)} \sum_{m=1}^{\frac{n}{2}} c_{-m} \frac{s^{2m}}{m!} \left( \sum_{r=1}^{m-1} \frac{1}{r} - 2 \sum_{r=1}^{2m-1} \frac{1}{r} \right) \right),$$

where the coefficients  $c_k$  are defined by the asymptotic expansion

$$\operatorname{Tr} e^{-tA_d(\gamma, \sigma)^2} \stackrel{t \rightarrow 0}{\sim} \sum_{k=-\frac{n}{2}}^{\infty} c_k t^k.$$

Hence, (see, [4, pp. 120–122])

$$\begin{aligned} Z_{S,\chi}(-s, \sigma) &= Z_{S,\chi}(s, \sigma) \cdot \left( \frac{\det(A_d(\gamma, \sigma) - s)}{\det(A_d(\gamma, \sigma) + s)} \right)^{-\frac{2 \dim(\chi)\chi(Y)}{\chi(X_d)}} = \\ &= Z_{S,\chi}(s, \sigma) \cdot \left( \frac{D^+(s)}{D^-(s)} \right)^{-\frac{2 \dim(\chi)\chi(Y)}{\chi(X_d)}} = \end{aligned}$$

$$Z_{S,\chi}(s, \sigma) \cdot \exp \left( -\frac{\pi}{T} \int_0^s P_\sigma(w) \left\{ \begin{array}{cc} \tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = \frac{1}{2} \\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = 0 \end{array} \right\} dw \right)^{-\frac{2 \dim(\chi)\chi(Y)}{\chi(X_d)}}.$$

This completes the proof.  $\square$

**Lemma 4.2.** *If  $\gamma$  is  $\sigma$ -admissible, then*

$$P_\sigma(w) = \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} w^{n-2k-1},$$

where

$$p_{n-2k-1} = \frac{2T}{\left(\frac{n}{2} - k - 1\right)!} c_{-(\frac{n}{2}-k)}, \quad k = 0, 1, \dots, \frac{n}{2} - 1,$$

$$c_{-\frac{n}{2}} = \frac{\left(\frac{n}{2} - 1\right)!}{2T}.$$

*Proof.* By [4, pp. 47-48],  $P_\sigma(0) = 0$ ,  $P_\sigma(-w) = -P_\sigma(w)$  and  $P_\sigma(w) = w \cdot Q_\sigma(w)$ , where  $Q_\sigma$  is an even polynomial. Hence,  $P_\sigma$  is an odd polynomial. Moreover,  $P_\sigma(w) \in \mathbb{R}[w]$  is a monic polynomial of degree  $n-1$  (see, e.g., [3, pp. 17-19], [11, pp. 240-243]).

Put

$$P_\sigma(w) = \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} w^{n-2k-1}, \quad p_{n-1} = 1.$$

By [4, p. 118],  $Q_\sigma(w) = \sum_{k=0}^{\frac{n}{2}-1} q_{n-2k-2} w^{n-2k-2}$ , where  $q_{2i} = \frac{2T}{i!} c_{-(i+1)}$ ,  $i = 0, 1, \dots, \frac{n}{2} - 1$ . In other words,

$$p_{n-2k-1} = q_{n-2k-2} = \frac{2T}{\left(\frac{n}{2} - k - 1\right)!} c_{-(\frac{n}{2}-k)}, \quad k = 0, 1, \dots, \frac{n}{2} - 1.$$

This completes the proof.  $\square$

**Theorem 4.3.** *Suppose  $\sigma_1 < 0$  fixed. If  $\gamma$  is  $\sigma$ -admissible, then*

$$Z_{S,\chi}(\sigma_1 + it, \sigma) = f(t) e^{g(t)} \cdot Z_{S,\chi}(-\sigma_1 - it, \sigma), \quad |t| \rightarrow \infty,$$

where

$$f(t) = \exp \left( -\sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{|t|}{t} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} (-1)^l \sigma_1^{n-2k-2l} t^{2l} + O(1) \right),$$

$$g(t) = - \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{K}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} (-1)^l \sigma_1^{n-2k-2l+1} |t|^{2l-1}.$$

*Proof.* Suppose  $\epsilon_\sigma = \frac{1}{2}$ . By Lemma 4.2,

$$K \int_0^s P_\sigma(w) \tan\left(\frac{\pi w}{T}\right) dw = K \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \int_0^s w^{n-2k-1} \tan\left(\frac{\pi w}{T}\right) dw.$$

Here, we assume that the integration will be carried out along the line segment joining the origin to  $s$  (see, [9, p. 211]). As  $|t| \rightarrow \infty$ ,

$$\tan \pi(\sigma_1 + i t) = i \frac{t}{|t|} + O\left(e^{-2\pi|t|}\right).$$

Hence, at points on a vertical line away from the real axis, one has

$$\begin{aligned} (4.1) \quad & K \int_0^s P_\sigma(w) \tan\left(\frac{\pi w}{T}\right) dw = \\ & K \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \left(\frac{\sigma_1}{t} + i\right)^{n-2k} \frac{t}{|t|} i \frac{t^{n-2k}}{n-2k} + \\ & K \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \left(\frac{\sigma_1}{t} + i\right)^{n-2k} \int_0^t y^{n-2k-1} O\left(e^{-2\pi \frac{|y|}{T}}\right) dy = \\ & \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{n-2k} \binom{n-2k}{l} \sigma_1^{n-2k-l} t^l i^l + S = \\ & \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} \sigma_1^{n-2k-2l} t^{2l} i^{2l} + \\ & \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} \sigma_1^{n-2k-2l+1} t^{2l-1} i^{2l-1} + S = \\ & \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} (-1)^l \sigma_1^{n-2k-2l} t^{2l} + \\ & \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{K}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} (-1)^l \sigma_1^{n-2k-2l+1} |t|^{2l-1} + S. \end{aligned}$$

Assume  $t > 0$ . We have



$$\int_0^t y^{n-2k-1} O\left(e^{-2\pi\frac{|y|}{T}}\right) dy = O\left(\int_0^t y^{n-2k-1} e^{-2\pi\frac{y}{T}} dy\right).$$

Applying integration by parts  $n - 2k - 1$  times, one easily obtains that

$$\int_0^t y^{n-2k-1} e^{-2\pi\frac{y}{T}} dy = O(1).$$

Hence,

$$\int_0^t y^{n-2k-1} O\left(e^{-2\pi\frac{|y|}{T}}\right) dy = O(1).$$

If  $t < 0$ , then

$$\int_0^t y^{n-2k-1} O\left(e^{-2\pi\frac{|y|}{T}}\right) dy = \int_0^{-t} y^{n-2k-1} O\left(e^{-2\pi\frac{|y|}{T}}\right) dy = O(1).$$

We conclude,  $S = O(1)$ . Therefore, by (4.1),

$$(4.2) \quad -K \int_0^s P_\sigma(w) \tan\left(\frac{\pi w}{T}\right) dw =$$

$$- \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} (-1)^l \sigma_1^{n-2k-2l} t^{2l} -$$

$$\sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{K}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} (-1)^l \sigma_1^{n-2k-2l+1} |t|^{2l-1} + O(1).$$

Now, suppose  $\epsilon_\sigma = 0$ . It is not difficult to verify that

$$\cot \pi(\sigma_1 + i t) = -i \frac{t}{|t|} + O\left(e^{-2\pi|t|}\right)$$

as  $|t| \rightarrow \infty$ . We have

$$(4.3) \quad -K \int_0^s P_\sigma(w) \left(-\cot\left(\frac{\pi w}{T}\right)\right) dw = K \int_0^s P_\sigma(w) \cot\left(\frac{\pi w}{T}\right) dw =$$

$$K \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \int_0^s w^{n-2k-1} \cot\left(\frac{\pi w}{T}\right) dw =$$

$$\begin{aligned}
& -K \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \left( \frac{\sigma_1}{t} + i \right)^{n-2k} \frac{t}{|t|} i \frac{t^{n-2k}}{n-2k} + S = \\
& - \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} (-1)^l \sigma_1^{n-2k-2l} t^{2l} - \\
& \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{K}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} (-1)^l \sigma_1^{n-2k-2l+1} |t|^{2l-1} + O(1).
\end{aligned}$$

Combining (4.2) and (4.3), we finally obtain

$$\begin{aligned}
(4.4) \quad & -K \int_0^s P_\sigma(w) \left\{ \begin{array}{ll} \tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = \frac{1}{2} \\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = 0 \end{array} \right\} dw = \\
& - \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} (-1)^l \sigma_1^{n-2k-2l} t^{2l} - \\
& \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{K}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} (-1)^l \sigma_1^{n-2k-2l+1} |t|^{2l-1} + O(1).
\end{aligned}$$

The theorem now follows from (4.4) and Lemma 4.1.  $\square$

**Lemma 4.4.** *If  $\gamma$  is  $\sigma$ -admissible, then*

$$|Z_S^i(\sigma_1 + i t)| = e^{O(|t|^{n-1})}$$

*uniformly in any bounded strip  $b_1 \leq \sigma_1 \leq b_2$  for  $i = 1, 2$ .*

*Proof.* Let  $c > \max\{\rho, |b_1|, |b_2|\}$ . It is enough to prove the assertion for a wider strip  $-c \leq \sigma_1 \leq c$ .

By (3.3),  $Z_S^1(s)$  and  $Z_S^2(s)$  are of order at most  $n$ . Furthermore, the infinite product (3.1) defining  $Z_{S,\chi}(s, \sigma)$  converges for  $\operatorname{Re}(s) > \rho$  (see, [4, pp. 98-99]). Hence, the lemma is implied by the Phragmén-Lindölef theorem and Theorem 4.3.  $\square$

**Theorem 4.5.** *If  $\gamma$  is  $\sigma$ -admissible, then*

$$N(t) = \frac{K}{2\pi} \sum_{k=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-k} p_{n-2k-1} \frac{t^{n-2k}}{n-2k} + \frac{1}{\pi} S(t) + O(1),$$

*where  $N(t)$  denotes the number of singularities of  $Z_{S,\chi}(s, \sigma)$  on the interval  $ix$ ,  $0 < x < t$  and  $S(t)$  is the variation of the argument of  $Z_{S,\chi}(s, \sigma)$  along*

*C.* Here,  $C$  denotes the portion of  $\partial R$  consisting of the vertical segment from  $a$  to  $a + it$  plus horizontal segment from  $a + it$  to  $it$ , where  $R$  is the rectangle defined by the inequalities  $-a \leq \operatorname{Re}(s) \leq a$ ,  $-t \leq \operatorname{Im}(s) \leq t$  for some  $a > \rho$ .

*Proof.* We follow [9].

Define  $\xi(s) = (Z_{S,\chi}(s, \sigma))^2 e^{\phi(s)}$ , where

$$\phi(s) = K \int_0^s P_\sigma(w) \left\{ \begin{array}{ll} \tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = \frac{1}{2} \\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma = 0 \end{array} \right\} dw.$$

Here, as earlier, we specify  $\phi(s)$  in the open upper and lower half-planes to be the value obtained by carrying out the integration along the line segment joining the origin to  $s$ . Furthermore, if  $\epsilon_\sigma = \frac{1}{2}$  resp.  $\epsilon_\sigma = 0$  and  $s$  is on the real line,  $s \neq \pm \frac{T}{2}, \pm \frac{3T}{2}, \pm \frac{5T}{2}, \dots$  resp.  $s \neq 0, \pm T, \pm 2T, \dots$ , we define  $\phi(s)$  by the requirement of continuity as  $s$  is approached from the upper half-plane.

By Lemma 4.1,

$$Z_{S,\chi}(-s, \sigma) = e^{\phi(s)} Z_{S,\chi}(s, \sigma).$$

Hence,

$$\xi(-s) = (Z_{S,\chi}(-s, \sigma))^2 e^{\phi(-s)} =$$

$$(Z_{S,\chi}(s, \sigma))^2 e^{2\phi(s)} e^{-\phi(s)} = (Z_{S,\chi}(s, \sigma))^2 e^{\phi(s)} = \xi(s).$$

As usual,  $\xi(s)$  is real on the real axis and so  $\overline{\xi(s)} = \xi(\bar{s})$ .

Assume that  $t$  is selected so that no singularity of  $Z_{S,\chi}(s, \sigma)$  occurs on the boundary of  $R$ . We have,

$$N(t) = \frac{1}{4} \cdot \frac{1}{2\pi i} \int_{\partial R} \frac{\xi'(s)}{\xi(s)} ds - \frac{1}{2} N_0 = \frac{1}{4} \cdot \frac{1}{2\pi} \operatorname{Im} \left( \int_{\partial R} \frac{\xi'(s)}{\xi(s)} ds \right) - \frac{1}{2} N_0,$$

where  $N_0 = O(1)$  is the number of singularities of  $Z_{S,\chi}(s, \sigma)$  on the real line.

From the functional equation for  $\xi(s)$  and the fact that  $\overline{\xi(s)} = \xi(\bar{s})$ , it follows in the classical way that

$$N(t) = \frac{1}{2\pi} \operatorname{Im} \left( \int_C \frac{\xi'(s)}{\xi(s)} ds \right) + O(1).$$

By (4.4),

$$\phi(\sigma_1 + it) =$$

$$\sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{t}{|t|} \frac{K i}{n-2k} \sum_{l=0}^{\frac{n}{2}-k} \binom{n-2k}{2l} (-1)^l \sigma_1^{n-2k-2l} t^{2l} +$$

$$\sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} \frac{K}{n-2k} \sum_{l=1}^{\frac{n}{2}-k} \binom{n-2k}{2l-1} (-1)^l \sigma_1^{n-2k-2l+1} |t|^{2l-1} + O(1).$$

Now,

$$\frac{\xi'(s)}{\xi(s)} = \phi'(s) + 2 \frac{Z'_{S,\chi}(s, \sigma)}{Z_{S,\chi}(s, \sigma)},$$

so

$$N(t) = \frac{1}{2\pi} \operatorname{Im} \left( \int_C \phi'(s) ds \right) + \frac{1}{\pi} \operatorname{Im} \left( \int_C \frac{Z'_{S,\chi}(s, \sigma)}{Z_{S,\chi}(s, \sigma)} ds \right) + O(1) =$$

$$\frac{1}{2\pi} \operatorname{Im} (\phi(it) - \phi(a)) + \frac{1}{\pi} S(t) + O(1) =$$

$$\frac{1}{2\pi} \operatorname{Im} \phi(it) + \frac{1}{\pi} S(t) - \frac{1}{2\pi} \phi(a) + O(1) =$$

$$\frac{K}{2\pi} \sum_{k=0}^{\frac{n}{2}-1} (-1)^{\frac{n}{2}-k} p_{n-2k-1} \frac{t^{n-2k}}{n-2k} + \frac{1}{\pi} S(t) + O(1).$$

This completes the proof.  $\square$

**Lemma 4.6.** *If  $\gamma$  is  $\sigma$ -admissible, then*

$$S(t) = O(t^{n-1}).$$

*Proof.* First, we extend the definition of  $S(t)$  to those values of  $t$  for which  $it$  is a pole or zero of  $Z_{S,\chi}(s, \sigma)$  by defining it to be

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2} (S(t + \varepsilon) + S(t - \varepsilon)).$$

Then, we have

$$S(t) = h(t) + O(1),$$

where  $h(t)$  is the variation of the argument of  $Z_{S,\chi}(s, \sigma)$  along the segment from  $a + it$  to  $it$ .

Proceeding in accordance with a custom, one easily concludes that

$$(4.5) \quad h(t) = O \left( \int_{\partial S} \log |Z_{S,\chi}(w + it, \sigma)| + \log |Z_{S,\chi}(w - it, \sigma)| dw \right) =$$

$$O \left( \sum_{i=1,2} \int_{\partial S} \log |Z_S^i(w + it)| dw + \int_{\partial S} \log |Z_S^i(w - it)| dw \right),$$

where  $S$  is the closed disc, centered at  $a$ , of radius  $a + \frac{1}{4}$ .

Now, the assertion follows from Lemma 4.4 and (4.5).  $\square$

**Corollary 4.7.** *If  $\gamma$  is  $\sigma$ -admissible, then*

$$N(t) = \frac{\dim(\chi) \operatorname{vol}(Y)}{nT \operatorname{vol}(X_d)} t^n + O(t^{n-1}).$$

*Proof.* An immediate consequence of (2.1), Theorem 4.5 and Lemma 4.6.  $\square$

**Corollary 4.8.** *Let  $-\rho \leq a \leq b \leq \rho$ . If  $\gamma$  is  $\sigma$ -admissible, then there exists a constant  $C$  such that*

$$N_R(t) = Ct^n + O(t^{n-1}),$$

where  $N_R(t)$  denotes the number of singularities of  $Z_{R,\chi}(s, \sigma)$  in the rectangle  $a \leq \operatorname{Re}(s) \leq b$ ,  $0 < \operatorname{Im}(s) < t$ .

*Proof.* Trivial consequence of the formula (3.2) and Corollary 4.7.  $\square$

**Remark 4.9.** Precise estimates of the number of singularities of the zeta functions represent an important tool which plays the key role in achieving more refined error terms in the prime geodesic theorem (see, [8], [1], [7]). Such counting functions may also be used elsewhere.

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## REFERENCES

- [1] M. Avdispahić and Dž. Gušić, On the error term in the prime geodesic theorem, *Bull. Korean Math. Soc.* **49** (2012), 367–372.
- [2] M. Avdispahić and Dž. Gušić, Order of Selberg’s and Ruelle’s zeta functions for compact even-dimensional locally symmetric spaces, *J. Math. Anal. Appl.* **413** (2014), 525–531.
- [3] U. Bröcker, On Selberg’s zeta functions, topological zeros and determinant formulas. Preprint, SFB 288 Berlin, No. 114, 1994.
- [4] U. Bunke and M. Olbrich, *Selberg zeta and theta functions. A Differential Operator Approach*, Akademie Verlag, Berlin 1995.
- [5] D. Fried, The zeta functions of Ruelle and Selberg. I, *Ann. Sci. Ec. Norm. Sup.* **19** (1986), 491–517.

- [6] D. Hejhal, *The Selberg trace formula for  $\mathrm{PSL}(2, \mathbb{R})$* , Vol. I. Lecture Notes in Mathematics 548. Springer-Verlag, Berlin-Heidelberg, 1976.
- [7] J. Park, Ruelle zeta function and prime geodesic theorem for hyperbolic manifolds with cusps, in G. van Dijk, M. Wakayama (eds.), *Casimir force, Casimir operators and Riemann hypothesis*. de Gruyter, Berlin 2010, pp. 89–104.
- [8] B. Randol, On the asymptotic distribution of closed geodesics on compact Riemann surfaces, *Trans. Amer. Math. Soc.* **233** (1977), 241–247.
- [9] B. Randol, The Riemann hypothesis for Selberg’s zeta-function and the asymptotic behavior of eigenvalues of the Laplace operator, *Trans. Amer. Math. Soc.* **236** (1978), 209–223.
- [10] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc (N.S.)* **20** (1956), 47–87.
- [11] M. Wakayama, Zeta functions of Selberg’s type associated with homogeneous vector bundles, *Hiroshima Math. J.* **15** (1985), 235–295.

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